5-8 Graphing and Solving Quadratic Inequalities

Main Ideas

- Graph quadratic inequalities in two variables.
- Solve quadratic inequalities in one variable.

New Vocabulary

quadratic inequality

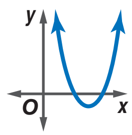
GET READY for the Lesson

Californian Jennifer Parilla is the only athlete from the United States to qualify for and compete in the Olympic trampoline event.

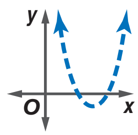
Suppose the height $h(t)$ in feet of a trampolinist above the ground during one bounce is modeled by the quadratic function $h(t) = -16t^2 + 42t + 3.75$. We can solve a quadratic inequality to determine how long this performer is more than a certain distance above the ground.

Graph Quadratic Inequalities You can graph **quadratic inequalities** in two variables using the same techniques you used to graph linear inequalities in two variables.

Step 1 Graph the related quadratic function, $y = ax^2 + bx + c$. Decide if the parabola should be solid or dashed.

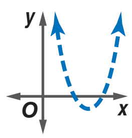


$\leq$ or $\geq$



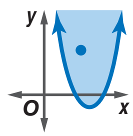
$<$ or $>$

Step 2 Test a point (x_1, y_1) inside the parabola. Check to see if this point is a solution of the inequality.

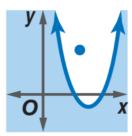


$y_1 \overset{?}{\geq} a(x_1)^2 + b(x_1) + c$

Step 3 If (x_1, y_1) is a solution, shade the region *inside* the parabola. If (x_1, y_1) is *not* a solution, shade the region *outside* the parabola.



(x_1, y_1) is a solution.



(x_1, y_1) is not a solution.

Study Tip

Look Back
For review of **graphing linear inequalities,** see Lesson 2-7.

EXAMPLE Graph a Quadratic Inequality

1 **Use a table to graph $y > -x^2 - 6x - 7$.**

Step 1 Graph the related quadratic function, $y = -x^2 - 6x - 7$.

Since the inequality symbol is $>$, the parabola should be dashed.

x	-5	-4	-3	-2	-1
y	-2	1	2	1	-2

Step 2 Test a point inside the parabola, such as $(-3, 0)$.

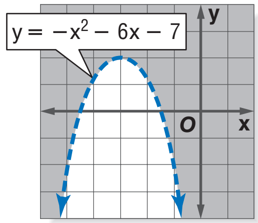
$y > -x^2 - 6x - 7$

$0 \overset{?}{>} -(-3)^2 - 6(-3) - 7$

$0 \overset{?}{>} -9 + 18 - 7$

$0 \overset{?}{>} 2$ ✗

So, $(-3, 0)$ is *not* a solution of the inequality.



Step 3 Shade the region outside the parabola.

CHECK Your Progress **Graph each inequality.**

1A. $y \le x^2 + 2x + 4$ **1B.** $y < -2x^2 + 3x + 5$

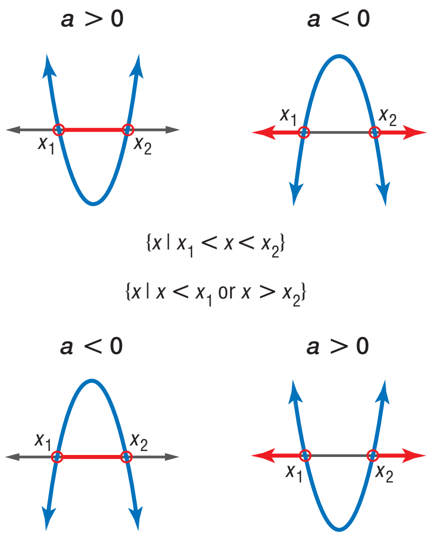
Solve Quadratic Inequalities To solve a quadratic inequality in one variable, you can use the graph of the related quadratic function.

To solve $ax^2 + bx + c < 0$, graph $y = ax^2 + bx + c$. Identify the x-values for which the graph lies *below* the x-axis.

For $\le$, include the x-intercepts in the solution.

To solve $ax^2 + bx + c > 0$, graph $y = ax^2 + bx + c$. Identify the x-values for which the graph lies *above* the x-axis.

For $\ge$, include the x-intercepts in the solution.



EXAMPLE Solve $ax^2 + bx + c > 0$

2 **Solve $x^2 + 2x - 3 > 0$ by graphing.**

The solution consists of the x-values for which the graph of the related quadratic function lies *above* the x-axis. Begin by finding the roots.

$x^2 + 2x - 3 = 0$	Related equation
$(x + 3)(x - 1) = 0$	Factor.
$x + 3 = 0$ or $x - 1 = 0$	Zero Product Property
$x = -3$ $\quad$ $x = 1$	Solve each equation.

(continued on the next page)

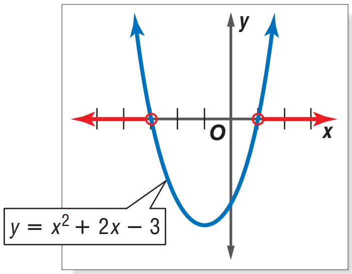
Study Tip

Solving Quadratic Inequalities by Graphing

A precise graph of the related quadratic function is not necessary since the zeros of the function were found algebraically.

Sketch the graph of a parabola that has x-intercepts at -3 and 1. The graph should open up since $a > 0$.

The graph lies above the x-axis to the left of $x = -3$ and to the right of $x = 1$. Therefore, the solution set is $\{x \mid x < -3 \text{ or } x > 1\}$.



CHECK Your Progress

Solve each inequality by graphing.

2A. $x^2 - 3x + 2 \geq 0$

2B. $0 \leq x^2 - 2x - 35$

EXAMPLE Solve $ax^2 + bx + c \leq 0$

3 **Solve $0 \geq 3x^2 - 7x - 1$ by graphing.**

This inequality can be rewritten as $3x^2 - 7x - 1 \leq 0$. The solution consists of the x-values for which the graph of the related quadratic function lies *on and below* the x-axis. Begin by finding the roots of the related equation.

$3x^2 - 7x - 1 = 0$ — Related equation

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ — Use the Quadratic Formula.

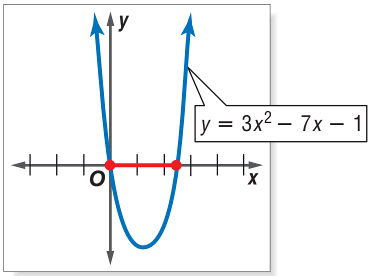
$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-1)}}{2(3)}$ — Replace a with 3, b with -7, and c with -1.

$x = \frac{7 + \sqrt{61}}{6}$ or $x = \frac{7 - \sqrt{61}}{6}$ — Simplify and write as two equations.

$x \approx 2.47$ $\quad x \approx -0.14$ — Simplify.

Sketch the graph of a parabola that has x-intercepts of 2.47 and -0.14. The graph should open up since $a > 0$.

The graph lies on and below the x-axis at $x = -0.14$ and $x = 2.47$ and between these two values. Therefore, the solution set of the inequality is approximately $\{x \mid -0.14 \leq x \leq 2.47\}$.



CHECK Test one value of x less than -0.14, one between -0.14 and 2.47, and one greater than 2.47 in the original inequality.

Test $x = -1$.	Test $x = 0$.	Test $x = 3$.
$0 \geq 3x^2 - 7x - 1$	$0 \geq 3x^2 - 7x - 1$	$0 \geq 3x^2 - 7x - 1$
$0 \overset{?}{\geq} 3(-1)^2 - 7(-1) - 1$	$0 \overset{?}{\geq} 3(0)^2 - 7(0) - 1$	$0 \overset{?}{\geq} 3(3)^2 - 7(3) - 1$
$0 \geq 9$ ✗	$0 \geq -1$ ✓	$0 \geq 5$ ✗

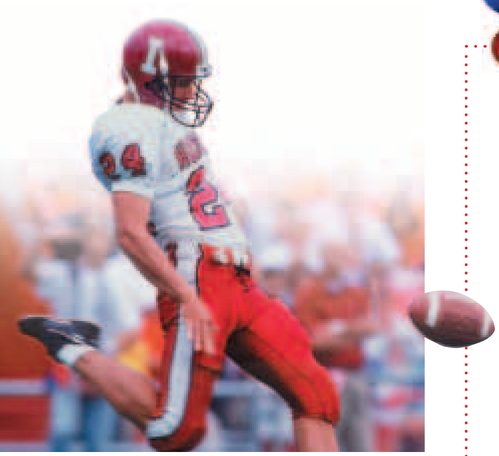
CHECK Your Progress

Solve each inequality by graphing.

3A. $0 > 2x^2 + 5x - 6$

3B. $5x^2 - 10x + 1 < 0$

Real-world problems that involve vertical motion can often be solved by using a quadratic inequality.



Real-World Link

A long hang time allows the kicking team time to provide good coverage on a punt return. The suggested hang time for high school and college punters is 4.5–4.6 seconds.

Source: www.takeaknee.com

Real-World EXAMPLE

4 **FOOTBALL The height of a punted football can be modeled by the function $H(x) = -4.9x^2 + 20x + 1$, where the height $H(x)$ is given in meters and the time x is in seconds. At what time in its flight is the ball within 5 meters of the ground?**

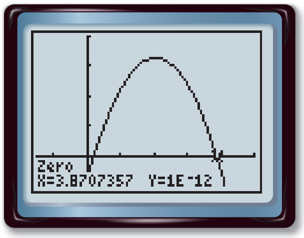
The function $H(x)$ describes the height of the football. Therefore, you want to find the values of x for which $H(x) \le 5$.

$H(x) \le 5$ Original inequality

$-4.9x^2 + 20x + 1 \le 5$ $H(x) = -4.9x^2 + 20x + 1$

$-4.9x^2 + 20x - 4 \le 0$ Subtract 5 from each side.

Graph the related function $y = -4.9x^2 + 20x - 4$ using a graphing calculator. The zeros of the function are about 0.21 and 3.87, and the graph lies below the x-axis when $x < 0.21$ or $x > 3.87$.



[−1.5, 5] scl: 1 by [−5, 20] scl: 5

Thus, the ball is within 5 meters of the ground for the first 0.21 second of its flight and again after 3.87 seconds until the ball hits the ground at 4.13 seconds.

CHECK The ball starts 1 meter above the ground, so $x < 0.21$ makes sense. Based on the given information, a punt stays in the air about 4.5 seconds. So, it is reasonable that the ball is back within 5 meters of the ground after 3.87 seconds.

CHECK Your Progress

4. Use the function $H(x)$ above to find at what time in its flight the ball is at least 7 meters above the ground.

Online **Personal Tutor at** algebra2.com

Study Tip

Solving Quadratic Inequalities Algebraically

As with linear inequalities, the solution set of a quadratic inequality is sometimes all real numbers or the empty set, ∅. The solution is all real numbers when all three test points satisfy the inequality. It is the empty set when none of the test points satisfy the inequality.

EXAMPLE Solve a Quadratic Inequality

5 **Solve $x^2 + x > 6$ algebraically.**

First solve the related quadratic equation $x^2 + x = 6$.

$x^2 + x = 6$ Related quadratic equation

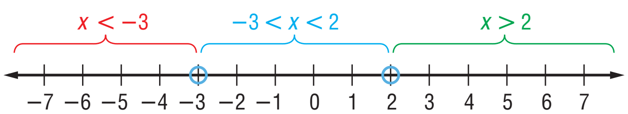
$x^2 + x - 6 = 0$ Subtract 6 from each side.

$(x + 3)(x - 2) = 0$ Factor.

$x + 3 = 0$ or $x - 2 = 0$ Zero Product Property

$x = -3$ $\quad x = 2$ Solve each equation.

Plot −3 and 2 on a number line. Use circles since these values are not solutions of the original inequality. Notice that the number line is now separated into three intervals.

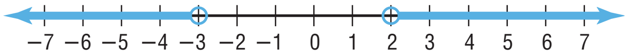


(continued on the next page)

Test a value in each interval to see if it satisfies the original inequality.

$x < -3$	$-3 < x < 2$	$x > 2$
Test $x = -4$.	Test $x = 0$.	Test $x = 4$.
$x^2 + x > 6$	$x^2 + x > 6$	$x^2 + x > 6$
$(-4)^2 + (-4) \stackrel{?}{>} 6$	$0^2 + 0 \stackrel{?}{>} 6$	$4^2 + 4 \stackrel{?}{>} 6$
$12 > 6$ ✓	$0 > 6$ ✗	$20 > 6$ ✓

The solution set is $\{x \mid x < -3 \text{ or } x > 2\}$. This is shown on the number line below.



CHECK Your Progress

Solve each inequality algebraically.

5A. $x^2 + 5x < -6$

5B. $x^2 + 11x + 30 \leq 0$

CHECK Your Understanding

Example 1 (p. 295)

Graph each inequality.

1. $y \geq x^2 - 10x + 25$

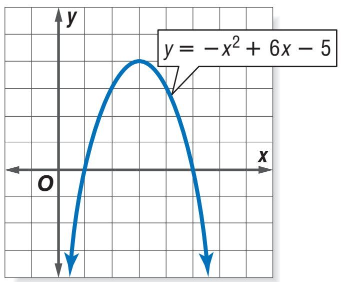
2. $y < x^2 - 16$

3. $y > -2x^2 - 4x + 3$

4. $y \leq -x^2 + 5x + 6$

Examples 2, 3 (pp. 295–296)

5. Use the graph of the related function of $-x^2 + 6x - 5 < 0$, which is shown at the right, to write the solutions of the inequality.



Examples 2, 3, 5 (pp. 295–298)

Solve each inequality using a graph, a table, or algebraically.

6. $x^2 - 6x - 7 < 0$

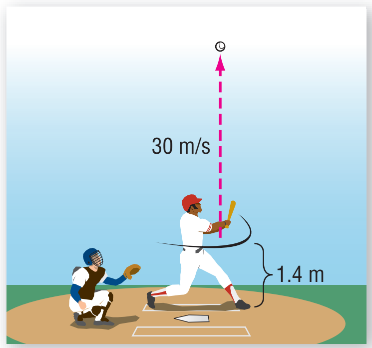
7. $x^2 - x - 12 > 0$

8. $x^2 < 10x - 25$

9. $x^2 \leq 3$

Example 4 (p. 297)

10. BASEBALL A baseball player hits a high pop-up with an initial upward velocity of 30 meters per second, 1.4 meters above the ground. The height $h(t)$ of the ball in meters t seconds after being hit is modeled by $h(t) = -4.9t^2 + 30t + 1.4$. How long does a player on the opposing team have to get under the ball if he catches it 1.7 meters above the ground? Does your answer seem reasonable? Explain.



Exercises

Graph each inequality.

11. $y \geq x^2 + 3x - 18$

12. $y < -x^2 + 7x + 8$

13. $y \leq x^2 + 4x + 4$

14. $y \leq x^2 + 4x$

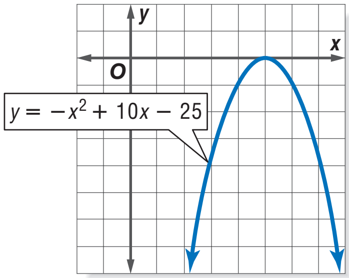
15. $y > x^2 - 36$

16. $y > x^2 + 6x + 5$

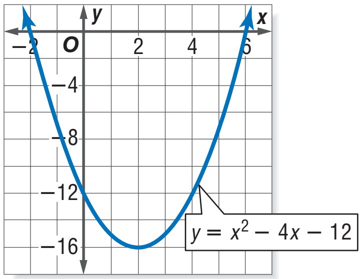
HOMEWORK HELP	
For Exercises	See Examples
11–16	1
17–20	2, 3
21–26	2, 3, 5
27, 28	4

Use the graph of the related function of each inequality to write its solutions.

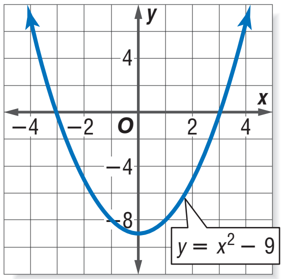
17. $-x^2 + 10x - 25 \geq 0$



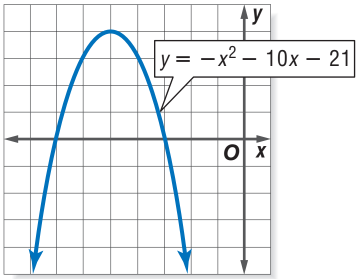
18. $x^2 - 4x - 12 \leq 0$



19. $x^2 - 9 > 0$



20. $-x^2 - 10x - 21 < 0$



Solve each inequality using a graph, a table, or algebraically.

21. $x^2 - 3x - 18 > 0$

22. $x^2 + 3x - 28 < 0$

23. $x^2 - 4x \leq 5$

24. $x^2 + 2x \geq 24$

25. $-x^2 - x + 12 \geq 0$

26. $-x^2 - 6x + 7 \leq 0$

27. LANDSCAPING Kinu wants to plant a garden and surround it with decorative stones. She has enough stones to enclose a rectangular garden with a perimeter of 68 feet, but she wants the garden to cover no more than 240 square feet. What could the width of her garden be?

28. GEOMETRY A rectangle is 6 centimeters longer than it is wide. Find the possible dimensions if the area of the rectangle is more than 216 square centimeters.

Real-World Career

Landscape Architect

Landscape architects design outdoor spaces so that they are not only functional, but beautiful and compatible with the natural environment.

Math Online

For more information, go to algebra2.com.

Graph each inequality.

29. $y \leq -x^2 - 3x + 10$

30. $y \geq -x^2 - 7x + 10$

31. $y > -x^2 + 10x - 23$

32. $y < -x^2 + 13x - 36$

33. $y < 2x^2 + 3x - 5$

34. $y \geq 2x^2 + x - 3$

Solve each inequality using a graph, a table, or algebraically.

35. $9x^2 - 6x + 1 \leq 0$

36. $4x^2 + 20x + 25 \geq 0$

37. $x^2 + 12x < -36$

38. $-x^2 + 14x - 49 \geq 0$

39. $18x - x^2 \leq 81$

40. $16x^2 + 9 < 24x$

41. $(x - 1)(x + 4)(x - 3) > 0$

42. BUSINESS A mall owner has determined that the relationship between monthly rent charged for store space r (in dollars per square foot) and monthly profit $P(r)$ (in thousands of dollars) can be approximated by the function $P(r) = -8.1r^2 + 46.9r - 38.2$. Solve each quadratic equation or inequality. Explain what each answer tells about the relationship between monthly rent and profit for this mall.

a. $-8.1r^2 + 46.9r - 38.2 = 0$

b. $-8.1r^2 + 46.9r - 38.2 > 0$

c. $-8.1r^2 + 46.9r - 38.2 > 10$

d. $-8.1r^2 + 46.9r - 38.2 < 10$

FUND-RAISING **For Exercises 43–45, use the following information.**
The girls' softball team is sponsoring a fund-raising trip to see a professional baseball game. They charter a 60-passenger bus for \$525. In order to make a profit, they will charge \$15 per person if all seats on the bus are sold, but for each empty seat, they will increase the price by \$1.50 per person.

43. Write a quadratic function giving the softball team's profit $P(n)$ from this fund-raiser as a function of the number of passengers n.

44. What is the minimum number of passengers needed in order for the softball team not to lose money?

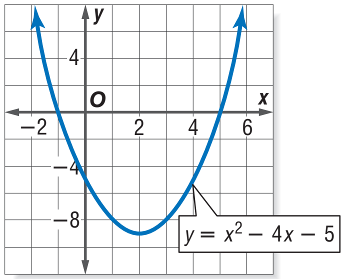
45. What is the maximum profit the team can make with this fund-raiser, and how many passengers will it take to achieve this maximum?

EXTRA PRACTICE
See pages 902, 930.

Math Online
Self-Check Quiz at
algebra2.com

H.O.T. Problems

46. REASONING Examine the graph of $y = x^2 - 4x - 5$.



a. What are the solutions of $0 = x^2 - 4x - 5$?

b. What are the solutions of $x^2 - 4x - 5 \geq 0$?

c. What are the solutions of $x^2 - 4x - 5 \leq 0$?

47. OPEN ENDED List three points you might test to find the solution of $(x + 3)(x - 5) < 0$.

48. CHALLENGE Graph the intersection of the graphs of $y \leq -x^2 + 4$ and $y \geq x^2 - 4$.

49. ***Writing in Math*** Use the information on page 294 to explain how you can find the time a trampolinist spends above a certain height. Include a quadratic inequality that describes the time the performer spends more than 10 feet above the ground, and two approaches to solving this quadratic inequality.

STANDARDIZED TEST PRACTICE

50. ACT/SAT If $(x + 1)(x - 2)$ is positive, which statement must be true?

A $x < -1$ or $x > 2$

B $x > -1$ or $x < 2$

C $-1 < x < 2$

D $-2 < x < 1$

51. REVIEW Which is the graph of $y = -3(x - 2)^2 + 1$?

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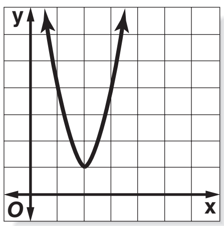

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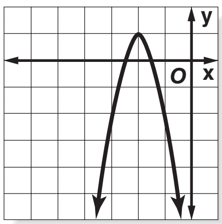

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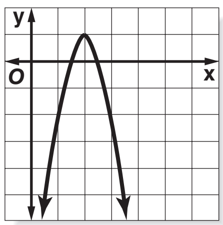

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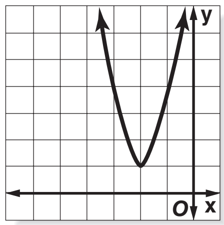


Spiral Review

Write each equation in vertex form. Then identify the vertex, axis of symmetry, and direction of opening. (Lesson 5-7)

52. $y = x^2 - 2x + 9$

53. $y = -2x^2 + 16x - 32$

54. $y = \frac{1}{2}x^2 + 6x + 18$

Solve each equation by using the method of your choice. Find exact solutions. (Lesson 5-6)

55. $x^2 + 12x + 32 = 0$

56. $x^2 + 7 = -5x$

57. $3x^2 + 6x - 2 = 3$

Solve each matrix equation or system of equations by using inverse matrices. (Lesson 4-8)

58. $\begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 18 \end{bmatrix}$

59. $\begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

60. $3j + 2k = 8$
$j - 7k = 18$

61. $5y + 2z = 11$
$10y - 4z = -2$

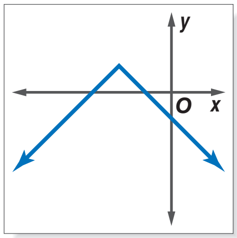
Find each product, if possible. (Lesson 4-3)

62. $\begin{bmatrix} -6 & 3 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ -3 & 6 \end{bmatrix}$

63. $\begin{bmatrix} 2 & -6 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -3 \\ 9 & 0 \\ -2 & 4 \end{bmatrix}$

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise. (Lesson 2-6)

64.



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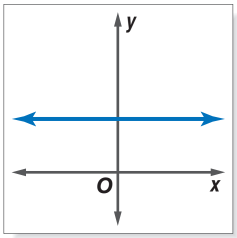
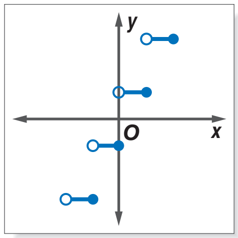


66.

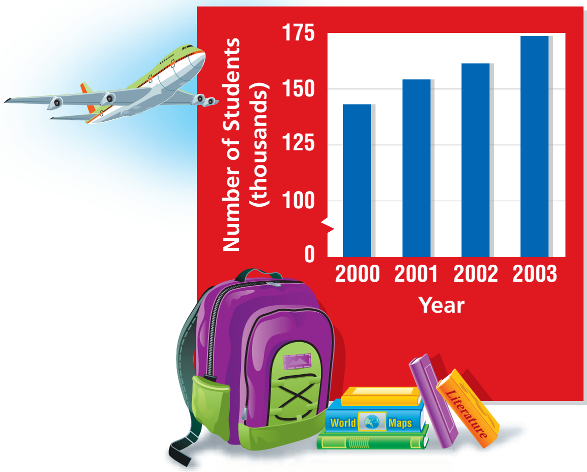


67. EDUCATION The number of U.S. college students studying abroad in 2003 increased by about 8.57% over the previous year. The graph shows the number of U.S. students in study-abroad programs. (Lesson 2-5)

a. Write a prediction equation from the data given.

b. Use your equation to predict the number of students in these programs in 2010.

Americans Study Abroad



Source: Institute of International Education

68. LAW ENFORCEMENT A certain laser device measures vehicle speed to within 3 miles per hour. If a vehicle's actual speed is 65 miles per hour, write and solve an absolute value equation to describe the range of speeds that might register on this device. (Lesson 1-6)